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Hilbert Space Representation of Time Evolution of Pure States

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Under a reasonable assumption it is shown that the set Tot_A of pure states of a physical system provided with the Scott information topology is homeomorphic to a subset of the Cantor discontinuum. Moreover, any "smooth" nonlinear time evolution with the set Tot_A as the phase space has a linear Hilber space representation.

1. INTRODUCTION

In Posiewnik (1985) I developed a mathematical language suitable for the description of physical systems with dynamics. My main demand was that the language should reflect a consistency between dynamics and structure. To this purpose I used category theory because in my opinion this theory allows one to treat the above problem in the most natural way. I consider category theory as a "language of structure" where all elements of the structure work together in a coherent way. In this respect I agree with the "doctrine" of Goguen *et al.* (1973): "Any species of mathematical structure is represented by a *category* whose objects "are of that structure" and whose *morphisms* "preserve" it.

A category may be thought of in the first instance as a universe for a particular kind of mathematical discourse. In this context one may ask how to choose the suitable category (universe of discourse) fitting the demands of the considered problem. The answer may be sought in category theory itself, because the theory allows one to construct the appropriate structures starting from very few general conditions. The realization of the objects of a category obtained in such a way by concrete sets equipped with some sort of structure is one of the last stages of our inquiry.

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In Posiewnik (1985) I used Scott's (1982) category of information systems and argued that it is a suitable category for a constructive description of the structure of physical situations. In the sequel I will denote the above category by the symbol IS.

In the choice of syntax I was guided by two main ideas. The first one may call ontological. To quote van Weizsäcker (1975), "But this description of what we, finite minds in time, can know would be meaningless if the reality which we know did not conform to this structure of our understanding. (One may suppose that in such a case we would not exist). Calling the facts we can know *information* we should expect that reality offers an information-like structure to our research."

It follows that similar constructions, perhaps on a very general level of abstraction, should be used in the semantics of description of physical situations as in the foundations of the mathematical theory of computation.

The second idea may be called pragmatic: If we try to develop a mathematical model of the real world in the spirit of constructivism, then the most important notion in this context is that of approximation.

The above ideas are implemented in a natural way in the structure of the category IS.

An extra bonus of the category of information systems is that the category is a Cartesian closed one, and so it can be interpreted as a theory of types, where objects (the sets of elements) represent types in the theory and moreover the higher types are of a very constructive nature. Elements of information systems of different types, e.g., elements representing states (methods of preparation), observables (methods of observation, measurements) and transmitters (operations on states), enter into the theory on an equal abstract footing (as elements of objects of the same category IS).

2. INFORMATION SYSTEMS

In Posiewnik (1985) I proposed to describe individual physical systems (quantum or classical) with the aid of structures called information systems. Information systems are already used in the mathematical theory of computation. I argued that in each theoretical description of the phenomenology of a preparation–observation process one can identify the set of all pure states of a physical system with the set of total elements of some information system.

For details concerning information systems, see Scott (1982) and Posiewnik (1985); for convenience I recall here some definitions and terminology.

An *information system* is a structure $(D, \Delta, \mathrm{Con}, \vdash)$. In our approach D is a set of properties of a physical system, and $\Delta \in D$ is the conjunction of all essential properties; Δ is the property engraved in each state of our system. Con is a family of finite subsets of D; if $u \in \text{Con}$, then the properties from u can be thought of without contradiction as "being together," so that they can be gathered together into "one thing." The symbol $\vdash$ denotes a binary relation between members of Con and members of D (we interpret the relation as a semantical relation of implication: if $u \in \text{Con}$ and $X \in D$, then $u \vdash X$ iff whenever the system has the properties belonging to u, then it has property X).

Some very simple and natural axioms are postulated for the family Con and relation $\vdash$ [for details see Posiewnik (1985)]. In most of the interpretations of the "logic" of physical systems one can easily point out the structure of an information system.

I argued that the (partial) state of one given physical system at time t_0 is uniquely determined by a set of properties engraved in the system by a preparation procedure and actual at that time t_0 (Piron, 1976). The pure state may be identified with the collection of *all* actual properties of the system. This is, indeed, precisely the role demanded of a pure state specification, namely to determine a maximally informative (maximally precise) *consistent* description of a physical system. Therefore, the subsets of properties that can be taken to define the states of a system should be consistent in themselves and deductively closed.

Definition. The states of a physical system represented by an information system $A = (D_A, \text{Con}_A, \Delta_A, \vdash_A)$ are those subsets x of D_A where: (i) all finite subsets of x are in Con_A, and (ii) whenever $u \subseteq x$ and $u \vdash_A X$, then $X \in x$.

The set of all such states is written as $|A|$.

A state that is not included in any strictly larger state from the set $|A|$ is called a *pure* state; the set of pure states is denoted by Tot_A.

In the theory of information systems we have a very natural notion of topology.

Definition (Scott, 1982). Let A be an information system. The *information topology* in the set A of all states is generated by a family of neighborhoods of the form

$$[u]_A = \{y \in |A| : u \subseteq y\}, \qquad \text{where } u \in \text{Con}_A$$

A neighborhood $[u]$ of a state $x \in |A|$ collects together all those states sharing the same (finite) amount of information $u \subseteq x$.

The main theorem concerning the topological structure of the set Tot_A of all pure states is the following:

Proposition 1 (Scott, 1982). The space Tot_A is a totally disconnected, compact Hausdorff space.

Now we shall prove:

Proposition 2. Let a physical system be described in terms of an information system $A = (D_A, \Delta_A, \mathrm{Con}_A, \vdash_A)$ such that the family Con_A is countable (a sufficient condition for that is f.e. the countability of D_A, which is not such a bold assumption; as a matter of fact, nothing in physics suggests noncountability).

Then the set Tot_A of pure states of our system is homeomorphic to a subset of the Cantor discontinuum.

Proof. Topology in the set $|A|$ of states is generated by the family $\{[u]_A, u \in \mathrm{Con}_A\}$ of neighborhoods. The neighborhoods are in a one-one correspondence with the elements of Con_A, so the space $|A|$ satisfies the second axiom of countability. Tot_A as a subspace of $|A|$ is second-countable, too. Every compact space is uniformizable; therefore, from Proposition 1, we see that the space Tot_A is uniformizable and second-countable and it follows (Semadeni, 1971) that Tot_A is a metrizable separable space.

The Cantor cube 2^ω consists of all sequences $a = (a_0, a_1, a_2, \ldots)$ of numbers 0 and 1, provided with the product topology.

If $a \in 2^\omega$, then

$$k = \sum_{n=0}^{\infty} \frac{2a_n}{3^{n+1}} = \varphi(a_0, a_1, \ldots) \tag{1}$$

is a number in the closed interval $I = [0, 1]$. The set l of all such numbers is called the *Cantor discontinuum.* It can be shown that the map φ: $2^\omega \to l$ given by (1) is a homeomorphism (the space I is equipped with the natural R^1 topology).

Every compact, totally disconnected metrizable space is homeomorphic to a subset of 2^ω (Semadeni, 1971), so from the above it is homeomorphic to a subset of the Cantor discontinuum l.

Tot_A is a compact, totally disconnected metrizable space, and this assertion ends the proof. □

It is rather obvious that states can be thought of as answers to numerous "yes-no" questions, i.e., as points of the Cantor cube. Therefore, the set Tot_A is isomorphic to a subset of the Cantor discontinuum. But we have shown something more, namely that this isomorphism is a topological one, i.e., the set of pure states can be conveniently embedded into the Cantor set $l \subset [0, 1]$ so as to preserve the topological structure. The states that are similar (near) in the information topology, i.e., the states that have many actual properties in common, are represented by points of the Cantor discontinuum that are near in R^1.

3. A REPRESENTATION THEOREM

Definition (de Vries, 1972). A *topological transformation group* (ttg) or a G-space is a triple (G, X, π) in which G is a topological group (the phase group), X is a topological space (the phase space), and π (the action of G on X) is a continuous function π: $G \times X \to X$ such that

$$\pi(e, x) = x, \qquad \pi(s, \pi(t, x)) = \pi(s \cdot t, x) \qquad \text{for all} \quad s, t \in G, \quad x \in X$$

(e denotes the unit of the group G). Let us assume that we are dealing with a concrete physical system represented in our formalism by an information system: $A = (D_A, \mathrm{Con}_A, \Delta_A, \vdash_A)$ and that the time evolution of the system is given by a topological transformation group (it is a relatively simple type of dynamics).

Because we want to describe a reversible time evolution of pure states of the system A, in our case the phase group G is simply the additive group R^1 and the phase space X is the space Tot_A equipped with the Scott's topology. It is rather a difficult task to investigate the properties of a general nonlinear topological transformation group, even in the R^1 case. For this reason we may try to find a suitable representation for our ttg (i.e., morphisms in the category of topological transformation groups) into special objects (f.e. linear transformation groups) whose structure is better understood.

In many cases similar to ours of time evolution of physical systems, the special objects that are targets for representation are related to Hilbert spaces. Let $L_2(R^1)$ denote the real Hilbert space consisting of all real-valued functions that are square-summable. Everything below can be done if one admits complex-valued functions as well.

Proposition 3. Let a physical system be represented by an information system $A = (D_A, \Delta_A, \mathrm{Con}_A, \vdash_A)$ with a countable family Con_A. Then any ttg $(R^1, \mathrm{Tot}_A, \pi)$ corresponding to a "smooth" time evolution of our system has a linear Hilbert space representation $(R^1, L_2(R^1), \rho)$.

Proof. We use the following theorem:

Theorem (de Vries, 1972). Let (R^1, X, σ) be a topological transformation group. If X is a compact, metrizable space and if the action of R^1 on X by σ is such that the set of invariant points in X [that is, the set $\{x \in X: \sigma(t, x) = x \text{ for all } t \in R^1\}$] is homeomorphic to a subset of R^1, then there is a topological embedding $\bar{\varphi}$: $X \to L_2(R^1)$ such that $(R^1, L_2(R^1), \rho)$ is a Hilbert space representation of ttg (R^1, X, σ), i.e.,

$$(\bar{\varphi} \cdot \sigma)(t, \cdot) = \rho(t, \bar{\varphi}(\cdot)) \qquad \text{for all} \qquad t \in R^1$$

The mapping $\bar{\varphi}$: $X \to L_2(R^1)$ is defined by the equality

$$(\bar{\varphi}(x))(t) \equiv f(t)\varphi(\sigma(t, x))$$

where $\varphi: X \to R^1$ is a bounded, continuous function, say $|\varphi(x)| \leq 1$ for all $x \in X$

$$f(t) = \exp(-|t|)$$

(this f is a proper weight function on R^1), and

$$[\rho(t, \tilde{x})](s) = \frac{\exp(-|s|)}{\exp(-|s+t|)} \tilde{x}(s+t) \quad \square$$

In our case the set Tot_A of pure states is homeomorphic to a subset of the Cantor discontinuum $l \subset I \subset R^1$; hence, the set of π-invariant states is homeomorphic to a subset of R^1. Moreover, Tot_A is a compact metrizable space; consequently, the thesis of de Vries' theorem holds for ttg (R^1, Tot_A, π).

Remarks. 1. The mapping $t \to \rho(t, \cdot)$ is a faithful representation of R^1 as a group of bounded, invertible, linear operators on the Hilbert space $L_2(R^1)$.

2. The group ρ does not depend on π. All information about the dynamics in the pure state space Tot_A is contained in the "wave" function $[\bar{\varphi}(x)](t)$, $x \in \mathrm{Tor}_A$.

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